4.1 I Theorem: Let f be a nonincreasing function on the bounded open interval (a, b). If f is bounded above on (a, b), then $\lim_{x \to a+} f(x)$ exists, while if f is bounded below on (a, b), then $\lim_{x \to b-} f(x)$ exists.

Proof case(i) Given f be a nonincreasing function

(ie) decreasing function

$\Rightarrow$ when $x < y$ $\quad f(x) > f(y)$

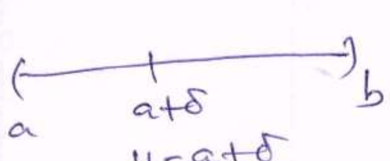
If f is bounded above on (a, b), therefore f has a l.u.b, let l.u.b $f(x) = M \quad \forall x (a, b)$

$$f(x) \leq M \quad \forall x \in (a, b)$$

$$f(x) < M + \varepsilon_1 \quad \forall x \in (a, b) \quad \text{—— (1)}$$

Since M is a l.u.b, therefore $M - \varepsilon_1$ is not an upper bound, therefore atleast one $y \in (a, b)$

Such that $f(y) > M - \varepsilon_1$



let $y = a + \delta$

$\therefore a < x < a + \delta$

$$f(x) > M - \varepsilon_1 \quad a < x < a + \delta \quad \text{—— (2)}$$

using (1) & (2)

$$M - \varepsilon_1 < f(x) < M + \varepsilon_1 \quad a < x < a + \delta$$

$$-\varepsilon_1 < f(x) - M < \varepsilon_1 \quad a < x < a + \delta$$

$$|f(x) - M| < \varepsilon_1 \quad a < x < a + \delta$$

$\Rightarrow \lim_{x \to a+} f(x) = M \Rightarrow \lim_{x \to a+} f(x)$ exists.

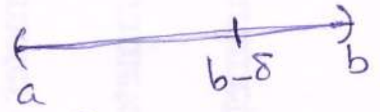
Case(ii) If f is bounded below on (a, b). ②

$\therefore$ f has g.l.b on (a, b)

let $\text{g.l.b } f(x) = L \quad \forall\, x \in (a, b)$

$$\therefore \quad f(x) \geq L \quad \forall\, x \in (a, b)$$

$$\therefore \quad f(x) > L - \varepsilon_1 \quad \forall\, x \in (a, b) \text{ --- } ③$$



Since L is the g.l.b of f

$L + \varepsilon_1$ is not a lower bound.

$\therefore$ there exists atleast one $y \in (a, b)$

such that $f(y) < L + \varepsilon_1$ let $y = b - \delta$

$$\therefore \text{ when } x \in (b - \delta, b)$$

$$f(x) < L + \varepsilon_1, \quad b - \delta < x < b \text{ --- } ④$$

$$\therefore \quad L - \varepsilon_1 < f(x) < L + \varepsilon_1 \qquad b - \delta < x < b$$

$$\Rightarrow \quad -\varepsilon_1 < f(x) - L < \varepsilon_1 \qquad b - \delta < x < b$$

$$|f(x) - L| < \varepsilon_1 \qquad b - \delta < x < b$$

$$\Rightarrow \quad \lim_{x \to b-} f(x) = L$$

$$\Rightarrow \quad \lim_{x \to b-} f(x) \text{ exists.}$$

4.1J COROLLARY. If f is a monotone function on the open interval (a, b) and if $c \in (a, b)$, then $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exists.

f is a monotone function

$\Rightarrow$ either f is increasing function or f is decreasing function.

let us assume f is increasing function.

choose $\delta > 0$ such that $(c-\delta, c+\delta)$ is contained in (a, b).

Then the values of f on the open interval $(c-\delta, c)$ are bounded above by $f(c)$ and hence f is bounded above on open interval $(c-\delta, c)$

$\therefore$ By theorem 4.1.H $\lim_{x \to c-} f(x)$ exists.

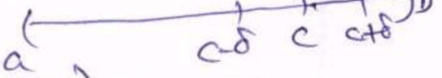
and the values of f on the open interval $(c, c+\delta)$ are bounded below by $f(c)$

$\therefore$ f is bounded below on $(c, c+\delta)$

$\therefore$ By thm 4.1H $\lim_{x \to c+} f(x)$ exists.

Suppose f is decreasing function on (a, b)

and $c \in (a, b)$



$\therefore$ f is decreasing function $(c-\delta, c+\delta)$

$\therefore$ f is bounded above on $(c, c+\delta)$ by $f(c)$

$\therefore$ by theorem 4.1 G $\lim_{x \to c+} f(x)$ exists.

and f is bounded below on $(c-\delta, c)$ by $f(c)$

$\therefore$ by theorem 4.1 G $\lim_{x \to c-} f(x)$ exists.

Hence proved.